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ON LOGARITHMIC POTENTIAL AND ANALYTIC FUNCTIONS.

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Two fields of mathematics which in most respects are so different that they might seem at first glance to have nothing in common are the Theory of Analytic Functions and the Theory of Potential. It may be shown, however, that they have much in common. It is not the object of this paper to present anything new in either field, but rather, (1) to consider the place occupied by Laplace's differential equation in both of them, (2) to set up and solve the problem of logarithmic potential for the most general distribution of distinct sources and sinks of finite strength in a given plane, and, (3) to point out the physical significance of the function

$$\log w,$$

when w is any rational analytic function of the complex variable.

§ 1. The Potential.

The potential V at a point P due to a system of n material particles was defined by Lagrange as follows:

$$V=\sum_{i=1}^{n}\frac{m_i}{v_i},$$

where m_i denotes the mass, and v_i the distance of the ith particle from the point P. Laplace showed that the potential function satisfies the partial differential equation

$$\triangle V\equiv\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

This equation is known as Laplace's equation. Poisson generalized Laplace's equation and showed that the potential function satisfies the equation

$$\triangle V = -4 \pi \rho,$$

where ρ is the density of the matter at the point P. If the n material particles form a continuous mass of finite extent bounded by the surface whose equation is $S(x', y', z') = 0$, Lagrange's definition of the potential function assumes the form

$$V = \int\int\int_s \frac{\gamma(x' y' z')\ dx'\ dy'\ dz'}{\sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]}},$$

$\gamma(x', y', z')$ being the density at the point (x', y', z'). If γ is a continuous function the integrand is a continuous function of the variables, x', y', z', x, y, and z, as is evident from the form it assumes in polar coordinates, viz:

$$V = \int\int\int \gamma\ r \sin \theta\ dr\ d\theta\ d\phi.$$

Since the partial derivatives of the integrand with respect to x, y, and z exist one may obtain

$$\frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial y} \text{ and } \frac{\partial V}{\partial z}$$

by differentiation under the integral signs. Thus

$$-\frac{\partial V}{\partial x} = \int\int\int_s \frac{\gamma(x', y', z') dx'\ dy'\ dz'}{r^2} \cdot \frac{(x'-x)}{r}.$$

But this integral is recognized as the x-component of the attraction of the system for a unit particle situated at the point P. Hence the relations between the potential function and the attraction at any point are as follows:

$$-\frac{\partial V}{\partial x} = X, \quad -\frac{\partial V}{\partial y} = Y, \quad -\frac{\partial V}{\partial z},$$

and the resultant,

$$F=+\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2}=+\sqrt{X^2+Y^2+Z^2},$$

acts in space so that the direction cosines are

$$\cos\alpha=\frac{-\frac{\partial V}{\partial x}}{F},\quad \cos\beta=\frac{-\frac{\partial V}{\partial y}}{F},\quad \cos\gamma=\frac{-\frac{\partial V}{\partial z}}{F}.$$

Thus, if the potential function for a system of particles is known the attraction is completely defined and the components of the attraction in any given direction may be computed at once.

§ 2. Laplace's Equation for Two Variables.

Solutions of Laplace's equation are called *harmonic functions*. Among the harmonic functions are those which depend upon two variables only. If $V\ (x,\ y)$ is a potential function the corresponding distribution of matter must be such that the z-component of attraction vanishes at every point. This may be realized in three ways:

1). Consider a thin uniform rod of finite length perpendicular to the x-y plane and bisected by it. At every point in the x-y plane the z-component of attraction is zero and the potential function will be a function of x and y alone. From the definition of potential it is clear that the number of rods of the kind described may be increased indefinitely and the number of variables in the potential function will not be changed.

2). If the rods are conceived to extend indefinitely in both directions parallel to the z-axis the potential function depends upon x and y only for all points in space since the z-component of attraction is zero for every point in space.

3). Suppose that the z-component of attraction due to a material system becomes neutralized by an equal opposing force at every point in the x-y plane. These forces become lost forces in the sense used by D'Alembert in his "Traité de Dynamique." The effective forces must then act along curves which lie in the x-y plane. The potential function for the effective forces in the x-y plane is therefore a harmonic function of the two variables x and y. For example, consider the x-y plane as a face of a finite smooth slab of metal lying in a horizontal position. If at time t_0 water stands on the plane, the depth varying from point to point, the downward pressure of the water at each point is counteracted by the opposing upward pressure of the plane. The difference of pressure of water at the x-y plane will cause motions of the water particles over the x-y plane from points of higher to points of lower pressure. If this instantaneous condition can be perpetuated by locating sources and sinks of appropriate strength at proper

positions in the x-y plane so-called *steady-streaming* will occur. If the metallic plate is thin and if it conducts electricity the positive and negative electrodes of any number of cells may be stationed at arbitrary points on the plane. Steady streaming of electricity will begin and continue as long as the electrodes are not moved and the relative potentials of the electrodes are not changed. The form of the potential function for two variable will be considered next.

§ 3. Logarithmic Potential.

Let $t(x, y)$ be a function which may or may not be harmonic. Let it be assumed that, if t is not harmonic, there is a function of t that is harmonic. The necessary condition which t must fulfill may be found as follows:

Differentiating $f(t)$,

$$\frac{\partial f(t)}{\partial x}=f'(t).\frac{\partial t}{\partial x}$$

$$\frac{\partial^2 f(t)}{\partial x^2}=f''(t)\left(\frac{\partial t}{\partial x}\right)^2+f'(t)\frac{\partial^2 t}{\partial x^2}.$$

Likewise, $$\frac{\partial^2 f(t)}{\partial y^2}=f''(t)\left(\frac{\partial t}{\partial y}\right)^2+f'(t)\frac{\partial^2 t}{\partial y^2}.$$

Adding and using the fact that $f(t)$ is harmonic,

$$\triangle f(t)=\left[\left(\frac{\partial t}{\partial x}\right)^2+\left(\frac{\partial t}{\partial y}\right)^2\right]f''(t)+\left[\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}\right]f'(t)=0.$$

Therefore, $$\frac{\dfrac{\partial^2 t}{\partial x^2}+\dfrac{\partial^2 t}{\partial x^2}}{\left(\dfrac{\partial t}{\partial x}\right)^2+\left(\dfrac{\partial t}{\partial x}\right)^2}=-\frac{f''(t)}{f'(t)}.$$

For brevity let L denote the differential operator in the left member of their relation. The right hand member is a function of t alone. It is evident then that the necessary condition which t must satisfy in order that there may be found a harmonic function of t is that $L\,t$ shall be a function of t alone, say $Q(t)$. When t satisfies this condition the required harmonic function $f(t)$ may be found by integration in the form

$$f(t)=k_1\int e^{-\int Qdt}\,d\,t+k_2.$$

Suppose, for example, that $t=x^3+y^3$. Then

$$Lt=\frac{2\,(x+y)}{3\,(x^4+y^4)}.$$

But this is not a function of the argument t alone, consequently there is no harmonic function of x^3+y^3.

Again, consider $\qquad t=(x^2+y^2)^{\frac{1}{2}}=r.$

Then $\qquad Lt=\frac{1}{t}=Q(t).$

Hence, $\qquad f(t)=k_1\int e^{-\int dt/t}\,d\,t+k_2=k_1\log t+k_2.$

Remembering that the potential function involves mass and distance, it is clear that so long as the mass elements of the system remain constant and their relative positions remain fixed, the potential function must vary only with r. But the form of the potential function of r only has been shown to be

$$V=k_1\log r+k_2,$$

the constants k_1 and k_2 depending upon the mass and potential units respectively. From its form the potential function of two variables is called *Logarithmic Potential.*

§ 4. Conjugate Functions.

The equation $V(x,\,y)=c_1$ defines for each value of c_1 a curve in the x-y plane. The set of curves $U(x\;y)=c_2$ which are orthogonal to the first set must satisfy the equation

$$\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\cdot\frac{\partial U}{\partial y}=0,$$

which is the necessary and sufficient condition for orthogonal functions.*

*Scheffer's "Auwendung der Differential und Integral Rechuung auf Geometrie," Vol. I, p. 110.

Cauchy and Riemann found as the necessary and sufficient condition that $W=V+iU$ should be an analytic function of the complex variable $z=x+iy$ the following differential equations:

$$\frac{\partial V}{\partial x}=\frac{\partial U}{\partial y} \text{ and } \frac{\partial V}{\partial y}=-\frac{\partial U}{\partial x}.$$

From these equations the condition for orthogonality follows at once. $U(x\,y)$ is said to be the conjugate function of the function $V(x, y)$. It is seen that if U is conjugate of V, then $-V$ will be conjugate of U, but in either case U and V are orthogonal functions. If U is orthogonal to V for every point within a given region of the $x\,y$ plane it is possible to choose the sign of U so that

$$V+i\,U=W$$

shall be an analytic function of z within the same region of the $x\,y$ plane.

The attraction due to a system of particles has already been given for the general case. For such distributions as lead to potential functions of two variables only, the magnitude of the attraction is given by the expression

$$F=\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2},$$

and its direction referred to the positive x-axis is given for each point by the expression

$$\tan\theta=\frac{\frac{\partial V}{\partial y}}{\frac{\partial V}{\partial x}}.$$

The slope of the tangent to the equipotential curves, $V=c_1$, is given by the expression

$$\tan\theta_1=-\frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial y}}.$$

Hence, $$|\theta_1-\theta|=\frac{\pi}{2}.$$

Thus the force acts along the normal to the equipotential curve at each point. For each distribution of matter there correspond two sets of functions which are mutually orthogonal and which if taken with proper signs are conjugate, one of the other. These functions are the potential function and the function which defines the stream lines. Conjugate functions are of fundamental importance in the theory of logarithmic potential as well as in the theory of analytic functions. Laplace's equation is the necessary condition in both fields. Riemann founded his theory of functions on this equation, and both he and Klein* drew many of their theorems on analytic functions from the laws and phenomena of conservative forces and steady currents. Holzmüller† has written several articles on various phases of this Physico-Function-theoretic unity. In at least two of these articles he deals with families of Cassinian ovals of higher order and with their orthogonal curves which he calls hyperbolas of higher order. It makes these higher plane curves much more interesting to find that they have an important physical significance.

§5. ANALYTIC FUNCTIONS AND STEADY STREAMING IN A PLANE.

As already shown the potential function for a thin rod is given by

$$v=k_1 \log r+k_2,$$

or, briefly, $$v=k_1 \log r,$$

where k_1 depends upon the density of the rod and the universal gravitational constant. The conjugate function is

$$U=k_1\theta=k_1 \text{ arc } \tan\frac{y}{x}.$$

Then $$V+i\,U=k_1 \log r+i\,k_1\theta=k_1 \log z.$$

So the simplest case of logarithmic potential leads to a simple logarithmic function on the function-theoretic side. Let the units be so chosen that

*Klein: On Riemann's Theory of Algebraic Functions.

†Isogonal verwandtschaften; Ueber einen Satz der Functionentheorie und seine Anwendung auf isothermische Kurven-systeme und auf einige Theorie der Mathematische Physik. Zeitschrift für Math. und Physik, Vol. 42, p. 217. Ueber die Abbildung und die lemniscatischen Coordinaten nter Ordnug. Crelle, 83, p. 38; Zusammenhang der Hyperbeln und Lemniscaten hoeheren Ordunng mit dem Ausgangpunkte der Funktionen Theorie. Zeitschrift für Math. und Physik, Vol. 29, p. 120.

$k_1=1$ if density equals one,
$k_1=2$ if density equals two, etc.

Then if k_1 is the density of a rod whose axis passes through the $x\ y$ plane at the point $(x_1\ y_1)$ the potential function is

$$V=k_1 \log r_1=k_1 \log\sqrt{(x-x_1)^2+(y-y_1)^2},$$

and the stream function is

$$U=k_1\ \theta_1=k_1 \text{ arc tan } \frac{y-y_1}{x-x_1}.$$

The corresponding analytic function is

$$W=k_1 \log (z-\alpha), \quad \alpha=x_1+i\,y_1.$$

Potential is a scalar quantity and hence the potential due to a set of n rods with densities $k_1, k_2 \ldots k_i \ldots k_n$, respectively, and passing through the x-y plane at the points $\alpha_1\equiv(x_1\ y_1), \ldots\ \alpha_i\equiv(x_i\ y_i), \ldots\ \alpha_n\equiv(x_n\ y_n)$ is given by

$$V=\sum_{i=1}^{n} k_i \log r_i=\log \prod_{i=1}^{n} r_i^{k_i}, \quad r_i=\sqrt{[(x-x_i)^2+(y-y_i)^2]}.$$

$$U=\sum_{i=1}^{n} k_i\ \theta_i, \text{ where } \theta_i=\text{arc tan } \frac{y-y_i}{x-x_i}.$$

Therefore $W=V+iU=\sum\limits_{i=1}^{n} k_i \log (z-\alpha_i)=\log \prod\limits_{i=1}^{n} (z-\alpha_i)^{k_i}.$

In the case of steady streaming of water or electricity over some portion of the $x\ y$ plane the potential function is

$$V=\sum_{i=1}^{n} k_i \log r_i-\sum_{j=1}^{m} h_j \log r_j,$$

where k_i and h_j are proportional to the capacities of the n sources and m sinks respectively. Corresponding to this

$$U = \sum_{i=1}^{n} k_i\, \theta_i - \sum_{j=1}^{m} h_j\, \theta_j,$$

and the analytic function is

$$W = \log \frac{\prod_{i=1}^{n} (z-\alpha_i)^{k_i}}{\prod_{j=1}^{m} (z-\beta_j)^{h_j}}.$$

The sources and sinks are the roots and poles of the rational analytic function

$$\frac{\prod_{i=1}^{n} (z-\alpha_i)^{k_i}}{\prod_{j=1}^{m} (z-\beta_i)^{h_j}}.$$

Every rational analytic function can be put in this form, and since it is possible to station sources and sinks of appropriate relative capacity at the points α_i and β_j there is a one-to-one correspondence between the rational analytic functions and the potential functions for a finite number of sources and sinks of finite capacity. It is therefore possible to write down at once the analytic function which will define and characterize the streaming whenever the location and strength of the various sources and sinks in the $x\, y$ plane is known. By means of limits one may solve many other examples in logarithmic potential. The equipotential curves

$$\frac{r_1^{k_1}, r_2^{k_2} \ldots r_i^{k_i} \ldots r_n^{k_n}}{r_{n+1}^{h_1} . r_{n+2}^{h_2} \ldots r_{n+j}^{h_j} \ldots r_{n+m}^{h_m}} = C_1,$$

are the Cassinian ovals of fractional order. Their foci are the points α_i and β_j, $i=1, 2, 3, \ldots n$, $j=1, 2, \ldots m$. The stream lines form the set of hyperbolas of higher order the polar equation for which is

$$k_1\, \theta_{k_1} + k_2\, \theta_{k_2} + \ldots + k_n\, \theta_{k_n} - h_1\, \theta_{h_1} - h_2\, \theta_{h_2} - \ldots - h_m\, \theta_{h_m} = C_2.$$

Each branch of one of these hyperbolas joins a source with some sink, the point at infinity counting as a sink when the streaming is steady and the sources have a greater total capacity than the sinks in the finite part of the plane.